THE GRONE MERRIS CONJECTURE AND
A QUADRATIC EIGENVALUE PROBLEM

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§0 Introduction

Given a graph $G = (V, E)$, we define the transpose degree sequence d_j^T to be equal to the number of vertices of degree at least j. We define L_G, the graph Laplacian, to be the matrix, whose rows and columns are indexed by the vertex set V, whose diagonal entry at v is the degree of v and whose value at a pair (v, w) is -1 if $(v, w) \in E$ and 0 otherwise.

Grone and Merris conjectured [GM]

Conjecture. *If $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of L_G in (weakly) decreasing order, then for any $1 \leq j \leq m$, we have*

$$\sum_{l=1}^{j} \lambda_l \leq \sum_{l=1}^{j} d_l^T.$$

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The first inequality is well known and the last inequality is indeed always an equality. On the class of threshold graphs, all the inequalities are equalities. The second inequality was proved by Duval and Reiner [DR], and this paper grew out of an attempt to understand their proof.

We will say that a graph is semi-bipartite if its vertex set is the union of a clique and an isolated set. We say a semi-bipartite graph is k regular, if every vertex in the isolated set has degree k. Duval and Reiner proved the second inequality by observing that it suffices to prove it for 1-regular semi-bipartite graph. They proved the second inequality by showing it was trivial for all but a few classes of 1 regular semi-bipartite graphs and then solving those cases one by one using a computer algebra system. However, if j is the number of vertices in the clique adjacent to some vertex in the isolated set, then it seems

that the jth inequality is quite difficult to prove and often fails to be an equality very narrowly. The jth inequality rather easily implies all the others.

The purpose of this paper is to prove

Main Theorem. *The Grone-Merris conjecture holds for 1-regular semibipartite graphs.*

The prood consists of an analysis of the roots of a certain polynomial. This polynomial arises from a quadratic eigenvalue problem (see [TM] for more information on QEP's) which is equivalent to finding the spectrum of the Laplacian. The analysis has two parts. The first part involves writing down lower degree polynomials each of which vanishes at a root of the original polynomial and to show that the lower degree polynomials are totally ordered by majorization. The second part is to use a homotopy to relate the order of root in the lower degree polynomials to the order of roots in the original polynomial.

We believe there is a good chance of extending the results of this paper to the setting of general semi-bipartite graphs. Every semi-bipartite graph gives rise to a polynomial eigenvalue problem in determining its spectrum. The associated lower degree polynomials are indeed totally ordered with respect to majorization. However some technical details may need to be worked out to ensure that the homotopy method works.

We add that semi-bipartite graphs are an extremely natural setting in which to study the Grone Merris conjecture. In particular, all threshold graphs are semi-bipartite. Also we have some intuition that edges towards higher degree vertices have more impact on the spectrum of the Laplacian than edges connecting low degree vertices. Therefore, we hope we are on a path leading towards resolution of the conjecture and we hope this paper gives rise to future work.

The paper is organized as follows: In section1, we state preliminary lemmas in linear algebra. In section 2, we derive our quadratic eigenvalue problem. In section 3, we give a couple of elementary examples which demonstrate the difficulty of the inequalities and illustrate our method. In section 4, we complete the analysis of the polynomial. In section 5, we do some bookkeeping which proves the main theorem.

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§1 Preliminaries

Throughout this paper, we will use the conventional notation that if $\lambda = (\lambda_1, \ldots, \lambda_j)$ and $\eta = (\eta_1, \ldots, \eta_j)$ are (weakly) decreasing vectors of real numbers then

$$\lambda \triangleright \eta,$$

whenever for any $1 \leq l \leq j$, we have that

$$\sum_{k=1}^{l} \lambda_k \geq \sum_{k=1}^{l} \eta_k.$$

We read this λ majorizes η.

We will abuse this notation as follows. If p and q are polynomials of the same degree with all real roots, we write $p \triangleright q$ provided that the roots of p, written in decreasing order, majorize the roots of q.

We will abuse the notation further. If A and B are matrices of the same dimension with all eigenvalues real, we will write $A \triangleright B$ provided that the eigenvalues of A written in decreasing order majorize the eigenvalues of B.

We recall a basic fact from linear algebra.

Proposition 1.1. *Let A be a symmetric $j \times j$ matrix. Let $\lambda_1, \ldots, \lambda_j$ be the eigenvalues in decreasing order. Then*

$$\sum_{k=1}^{l} \lambda_k \geq \mathrm{trace}(EAE),$$

where E is any orthogonal projection of rank l. The inequality is attained when E is the projection into the first l eigenvectors.

This has many important consequences. We state a few.

Proposition 1.2. *Let $\lambda_1, \ldots, \lambda_j$ be a decreasing sequence of real numbers. Let A be a diagonal matrix with the λ's along the diagonal.*

$$A = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda_j \end{pmatrix}.$$

Let B be a martix with zeroes along the diagonal and 1's everywhere else:

$$B = \begin{pmatrix} 0 & 1 & \ldots & 1 \\ 1 & 0 & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \ldots & 0 \end{pmatrix}.$$

Then

$$A + B \triangleright A,$$

and

$$A - B \triangleright A.$$

Proof. Let E_l be the orthogonal projection of a vector into it first l components.

$$\lambda_1 + \lambda_2 + \ldots \lambda_l = \mathrm{trace}(E_l(A + B)E_l) = \mathrm{trace}(E_l(A - B)E_l).$$

Now we apply Proposition 1.1 $\square$

Corollary 1.3. *Let a, b be positive real numbers with $b < a$. Then with A and B as above*

$$A - aB \rhd A - bB.$$

(Similarly, we have

$$A + aB \rhd A + bB.)$$

Proof. Let E be the l dimensional orthogonal projection into the l greatest eigenvalues of $A - bB$. Then

$$\text{trace}(E(A - bB)E) = \text{trace}(EAE) - b\,\text{trace}(EBE).$$

Applying the argument of Proposition 1.2, we see that $A - bB$ majorizes A so that since E achieves the maximum, it must be that $\text{trace}(EBE)$ is nonpositive. But

$$\text{trace}(E(A - aB)E) = \text{trace}(EAE) - a\,\text{trace}(EBE) \geq \text{trace}(E(A - bB)E).$$

Proposition 1.1 yields the desired result. $\square$

We will meet the matrix $A + B$ frequently in this paper. In the following proposition, we compute its determinant.

Proposition 1.4. *With A and B as above, and none of the λ_j equal to 1,*

$$\det(A + B) = (\prod_{l=1}^{j}(\lambda_l - 1))(1 + \sum_{l=1}^{j}\frac{1}{\lambda_l - 1}).$$

In the case that one or more of the λ_j's is equal to 1, the determinant may be read from the above by formal cancellation. If two or more of the λ_j's equal 1, the determinant is 0.

Proof.

$$A + B = A - I + (B + I).$$

Now just observe that $B + I$ has rank 1. The determinant is a multilinear antisymmetric function on the rows. Thus when we expand $\det(A - I + (B + I))$, any term involving more than one row of $B + I$ disappears. We are left with the expression above. $\square$

§2 GRAPH LAPLACIANS AND QUADRATIC EIGENVALUE PROBLEMS

Let $G = (V, E)$ be graph. For each vertex $v \in V$, let d_v be its degree.

We define its Laplacian L_G, to be a matrix whose rows and columns are indexed by v, and whose components are given by $(L_G)_{vv} = d_v$ while for $v, w \in V$ with $v \neq w$, we have

$$(L_G)_{vw} = -chi_E(v, w),$$

where χ_E is the indicator function of E, the set of edges. The Laplacian L_G is always positive definite since it is the matrix associated to the quadratic form

$$Q_G(x,x) = \sum_{(v,w)\in E} (x_v - x_w)^2.$$

We restrict to a special class of graphs. We say that a graph $G = (V, E)$ is semi-bipartite if V can be written as a disjoint union $V = V_g \cup V_b$ where V_g is a clique complete while V_b is an isolated set. We say a semi-bipartite graph $G = (V_g \cup V_b, E)$ is d-regular for a fixed positive integer d provided every vertex of V_b has degree d. From this point on, we will be considering a 1-regular semibipartite graph.

Following Duval and Reiner, we divide our vertices into various classes, permutations of which are automorphisms of the graph. We denote $\#(V_g) = n$. We denote by j, the number of vertices of V_g which are adjacent to some vertex of V_b. Clearly $j \leq n$. We consider the case when $j = n$ as degenerate, and we defer its consideration to §5. For now, we assume $j < n$. We denote by $v_1, \ldots, v_j$ those vertices of V_g which are adjacent to some vertex of V_b. We let $V_{g,extra}$ to be the set of those $n - j$ vertices of V_g not adjacent to any vertex in V_b. We denote by W_l, the set of vertices of V_b which are adjacent to the vertex v_l . We denote by k_l the cardinality of W_l.

We view L_G as acting on functions on V, and we are now in a position to identify several invariant subspaces of L_G. We say a function f has sum zero if $\sum_{v\in V} f(v) = 0$.

On any function supported on $V_{g,extra}$ having sum zero, the Laplacian L_G acts by multiplication by n. We have identified $n - j - 1$ eigenvalues equal to n. On any function supported on some W_l which has sum zero, the Laplacian L_G acts by multiplication by 1. We have identified $\#(V_b) - j$ eigenvalues equal to 1. To understand the spectrum of L_G, it now suffices to restrict to the space of functions which are constant on $V_{g,extra}$ and on each W_l. We describe an orthonormal basis for this space.

Here for $1 < l < j$,the vector e_l will be the function equal to 1 on v_l and zero elsewhere, e_{extra} the vector equal to $\frac{1}{\sqrt{n-j}}$ on $V_{g,extra}$ and zero elsewhere and f_l, the vector equal to $\frac{1}{\sqrt{k_l}}$ on C and zero elsewhere. We proceed to write down the $2j + 1$ dimensional matrix of L_G acting on this basis, which we will denote (considering j to be fixed) as $M(n, k_1, \ldots, k_j)$.

$$M(n, k_1, \ldots, k_j) =$$

$$\begin{pmatrix} n+k_1-1 & -1 & \ldots & -1 & -\sqrt{n-j} & -\sqrt{k_1} & 0 & \ldots & 0 \\ -1 & n+k_2-1 & \ldots & -1 & -\sqrt{n-j} & 0 & -\sqrt{k_2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & \ldots & n+k_j-1 & -\sqrt{n-j} & 0 & 0 & \ldots & -\sqrt{k_j} \\ -\sqrt{n-j} & -\sqrt{n-j} & \ldots & -\sqrt{n-j} & j & 0 & 0 & \ldots & 0 \\ -\sqrt{k_1} & 0 & \ldots & 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & -\sqrt{k_2} & \ldots & 0 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & -\sqrt{k_j} & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

Duval and Reiner studied the cases $j = 2$ and $j = 3$ using computer algebra. The remaining cases they avoided because they were only trying to prove the second inequality. We shall analyze this matrix directly.

First note that $M(n, k_1, \ldots, k_j)$ has one eigenvalue which is zero since the vector

$$(1, \ldots, 1, \sqrt{n-j}, \sqrt{k_1}, \ldots, \sqrt{k_j})$$

is in the kernel. We reduce the dimension by one writing a basis for the orthonormal complement to the kernel.

$$g_l = e_l - \frac{1}{\sqrt{n-j}} e_{extra},$$

and

$$h_l = f_l - \frac{\sqrt{k_l}}{\sqrt{n-j}} e_{extra}.$$

In this basis, the Laplacian L_G restricts to the matrix

$$N(n, k_1, \ldots, k_j) =$$

$$\begin{pmatrix} n+k_1 & 0 & \ldots & 0 & 0 & \sqrt{k_1} & \ldots & \sqrt{k_1} \\ 0 & n+k_2 & \ldots & & \sqrt{k_2} & 0 & \ldots & \sqrt{k_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & n+k_j & \sqrt{k_j} & \sqrt{k_j} & \ldots & 0 \\ -\sqrt{k_1} & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & -\sqrt{k_2} & \ldots & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & -\sqrt{k_j} & 0 & 0 & \ldots & 1 \end{pmatrix}$$

We take a less magnified view of this matrix:

$$N(n, k_1, \ldots, k_j) = \begin{pmatrix} A & B \\ C & I \end{pmatrix}.$$

Here A, B, and C are $j \times j$ matrices and I is the identity. Now if λ is an eigenvalue different from 1 of $N(n, k_1, \ldots, k_j)$, it must be we can find a pair v, w of vectors, at least one of which is nonzero so that

$$(A - \lambda)v + Bw = 0,$$

and

$$Cv + (I - \lambda)w = 0.$$

Solving for w in terms of v and observing that $I - \lambda$ commutes with everything, we obtain that the matrix

$$(A - \lambda)(I - \lambda) - BC$$

is non-invertible. This is the promised quadratic eigenvalue problem. In our case, we see that

$$(A - \lambda)(I - \lambda) - BC =$$

$$\begin{pmatrix} (n + k_1 - \lambda)(1 - \lambda) & k_2 & \ldots & k_j \\ k_1 & (n + k_2 - \lambda)(1 - \lambda) & \ldots & k_j \\ \vdots & \vdots & \vdots & \vdots \\ k_1 & k_2 & \ldots & (n + k_j - \lambda)(1 - \lambda) \end{pmatrix}$$

Dividing on the right by the matrix

$$\begin{pmatrix} k_1 & 0 & \ldots & 0 \\ 0 & k_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & k_j \end{pmatrix},$$

we see we must find the values where

$$\begin{pmatrix} \frac{(n+k_1-\lambda)(1-\lambda)-k_1}{k_1} & 0 & \ldots & 0 \\ 0 & \frac{(n+k_2-\lambda)(1-\lambda)-k_2}{k_2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \frac{(n+k_j-\lambda)(1-\lambda)-k_j}{\lambda} \end{pmatrix} + \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \end{pmatrix}.$$

We can take the determinant of the above sum using Proposition 1.4. We obtain the polynomial

$$F_{n, k_1, \ldots, k_j}(\lambda) = G_{n, k_1, \ldots k_j}(\lambda) \prod_{l=1}^{j} [(n + k_l - \lambda)(1 - \lambda) - k_l],$$

where

$$G_{n,k_1,...k_j}(\lambda) = 1 + \sum_{l=1}^{j} \frac{k_l}{(n + k_l - \lambda)(1 - \lambda) - k_l}.$$

The sum of the j largest roots of $F_{n,k_1,...,k_j}(\lambda)$ shall be the subject of sections 3 and 4.

§3 ILLUSTRATIVE EXAMPLES

Our goal in this section and the next is to bound the j largest roots of $F_{n,k_1,...,k_j}(\lambda)$ by

$$jn + \sum_{l=1}^{j} k_l.$$

This bound is the jth inequality in the Grone Merris conjecture.

In this section, we do a couple of simple examples. The first is intended to demonstrate that our inequality can seem very close to being tight even far from the threshold situation. The second is intended to illustrate the method we shall develop in §4.

Our first example is the case when the k's are all equal. That is

$$k_1 = k_2 = \cdots = k_j = k.$$

In that case, we can readily factor

$$F_{n,k_1,...,k_j}(\lambda) = ((n + k - \lambda)(1 - \lambda) - k)^{j-1}((n + k - \lambda)(1 - \lambda) + (j - 1)k).$$

Here, we can write out the j largest roots explicitly. We have a root with multiplicity $j - 1$ given by

$$r_1 = \frac{n + k + 1 + \sqrt{(n + k + 1)^2 - 4n}}{2},$$

and a root with multiplicity 1 given by

$$r_2 = \frac{n + k + 1 + \sqrt{(n + k + 1)^2 - 4n - 4jk}}{2}.$$

Observing that the square root function is concave, we see that

$$(j - 1)r_1 + r_2 \leq j\left(\frac{n + k + 1 + \sqrt{(n + k + 1)^2 - 4(n + k)}}{2}\right) = j(n + k),$$

which is exactly the desired inequality.

This shows that the inequality can be very tight: concavity of square roots is a second order effect. It also suggests that the Grone Merris conjecture might be viewed as a type of convexity inequality, although we have been unable to carry this out.

Our second example is one of those studied by Duval and Reiner using computer algebra. We show how to do it by first year calculus. The following section in which we prove the inequality in general is in fact a straightforward generalization of this approach.

We take the case $j = 2$ with k_1 and k_2 distinct. We see from the definition that

$$F_{n,k_1,k_2}(\lambda) = (1 - \lambda)^2(n + k_1 - \lambda)(n + k_2 - \lambda) - k_1 k_2.$$

Since F is an upward facing quartic, we see that since $F(n)$ is positive and $F(n + k_1)$ is negative, F has at least two roots greater than n. Since $F(1)$ is negative, we see that there are only two roots larger than n and these are the roots we must sum. Let us denote them in decreasing order, s_1 and s_2.

Now consider the quadratics

$$Q_1(\lambda) = (n + k_1 - \lambda)(n + k_2 - \lambda) - \frac{k_1 k_2}{(1 - s_1)^2}.$$

and

$$Q_2(\lambda) = (n + k_1 - \lambda)(n + k_2 - \lambda) - \frac{k_1 k_2}{(1 - s_2)^2}.$$

We see immediately that $Q_1(s_1) = Q_2(s_2) = 0$. Moreover, it is not hard to see that s_1 is the largest root of Q_1 and s_2 is the smallest root of Q_2. Let r_1 be the largest root of Q_2 and r_2 be smallest root of Q_1. This follows from the fact that $Q_1(n + k_1)$ and $Q_2(n + k_1)$ are both negative and s_1 is larger than $n + k_1$ while s_2 is smaller.

Now Q_1 and Q_2 differ by a constant. Since $s_1 > s_2$, we have that Q_1 is always larger than Q_2. There for $Q_1 \triangleright Q_2$. Thus $s_1 + s_2 < r_1 + s_2$. But the right hand side is the sum of the roots of Q_2 which we may read off from the formula.

$$r_1 + s_2 = 2n + k_1 + k_2.$$

This is the desired inequality.

§4 THE HEART OF THE MATTER

As before, we let $k_1 \geq k_2 \cdots \geq k_j \geq 1$ and $n > j \geq 2$. We define

$$G_{n,k_1,\ldots k_j}(\lambda) = 1 + \sum_{l=1}^{j} \frac{k_l}{(n + k_l - \lambda)(1 - \lambda) - k_l}.$$

Let
$$F_{n,k_1,\ldots,k_j}(\lambda) = G_{n,k_1,\ldots k_j}(\lambda) \prod_{l=1}^{j}[(n+k_l-\lambda)(1-\lambda)-k_l].$$

The function $F_{n,k_1,\ldots,k_j}(\lambda)$ is a polynomial of degree $2j$. Note that $F_{n,k_1,\ldots,k_j}(\lambda)$ is the determinant of the matrix

$$M_{n,k_1,\ldots,k_l} = \begin{pmatrix} (n+k_1-\lambda)(1-\lambda) & 1 & \cdots & 1 \\ 1 & (n+k_2-\lambda)(1-\lambda) & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & (n+k_j-\lambda)(1-\lambda) \end{pmatrix}.$$

The goal of this section is to prove:

Main Lemma. *The roots of the polynomial $F_{n,k_1,\ldots,k_j}(\lambda)$ are positive and real. Let us denote them by $s_1 \geq s_2 \geq \ldots s_j \geq s_{j+1} \geq \cdots \geq s_{2j}$. Then*

$$\sum_{l=1}^{j} s_l \leq jn + \sum_{l=1}^{j} k_l.$$

The fact that the roots are positive and real follows from the positivity of the matrix $M_{n,k_1,\ldots,k_l}$. However, we may also demonstrate it using the intermediate value theorem, in a way that localizes the roots and counts their multiplicity.

We define
$$r_l^+ = \frac{n+k_l+1+\sqrt{(n+k_l+1)^2-4n}}{2},$$
and
$$r_l^- = \frac{n+k_l+1-\sqrt{(n+k_l+1)^2-4n}}{2},$$
to be the roots of the quadratic $(n+k_l-\lambda)(1-\lambda)-k_l$. Notice that we have

$$0 < r_1^- \leq r_2^- \leq \ldots r_j^- < 1 < n+k_j \leq r_j^+ \leq r_{j-1}^+ \leq \cdots \leq r_1^+ < n+k_1+1.$$

It is evident that if k_l appears in the list $k_1,\ldots,k_j$ with multiplicity m then r_l^+ and r_l^- appear amongst the roots of $F_{n,k_1,\ldots,k_j}$ with multiplicity $m-1$. All remaining roots of $F_{n,k_1,\ldots,k_j}$ are zeroes of $G_{n,k_1,\ldots,k_j}$.

We note that $G_{n,k_1,\ldots,k_j}$ has singularities at each of the r_l^-'s and each of the r_l^+'s. The function $G_{n,k_1,\ldots,k_j}(\lambda)$ approaches $+\infty$ from the left and $-\infty$ from the right at each r_l^- and approaches $-\infty$ from the right and $+\infty$ from the left at each r_l^+. (The preceding is

because quadratics with distinct roots change signs at each of their roots.) Thus, by the intermediate value theorem, the function $G_{n,k_1,\ldots,k_j}(\lambda)$ has a zero between r_l^+ and r_{l-1}^+ for every $l > 1$ for which k_l and k_{l-1} are distinct and a zero between r_{l-1}^- and r_l^- for every $l > 1$ for which k_l and k_{l-1} are distinct. Moreover, $G_{n,k_1,\ldots,k_j}(n) = 1 - \frac{j}{n}$ which is positive. Since $G_{n,k_1,\ldots,k_j}$ approaches $-\infty$ from the right at r_j^- and from the left at r_j^+, there must be a zero of $G_{n,k_1,\ldots,k_j}$ between r_j^- and n and another zero between n and r_j^+. We have thus accounted for all the roots of $F_{n,k_1,\ldots,k_j}$ and indeed all the zeroes of $G_{n,k_1,\ldots,k_j}$. In particular, if we are interested in controlling the sum of the j largest roots of $F_{n,k_1,\ldots,k_j}$, we are interested precisely in the sum of those roots larger than n.

When λ is larger than n, the frequently occuring factor $(1-\lambda)$ is a large negative number whose order of maginitude varies slowly. Therefore, we "approximate" the factor $1 - \lambda$ by a constant. In other words, we introduce, for any constant $a < 1 - n$, the functions

$$G^a_{n,k_1,\ldots,k_j}(\lambda) = 1 + \sum_{l=1}^{j} \frac{k_l}{(n + k_l - \lambda)a - k_l}.$$

and

$$F^a_{n,k_1,\ldots,k_j}(\lambda) = G^a_{n,k_1,\ldots k_j}(\lambda) \prod_{l=1}^{j} [(n + k_l - \lambda)a - k_l].$$

We would like to be able to relate the organization of the roots of $F^a_{n,k_1,\ldots,k_j}(\lambda)$ to that of $F_{n,k_1,\ldots,k_j}(\lambda)$ so we introduce a homotopy between them. More precisely, we define

$$G^{a,t}_{n,k_1,\ldots,k_j}(\lambda) = 1 + \sum_{l=1}^{j} \frac{k_l}{(n + k_l - \lambda)(ta + (1 - t)(1 - \lambda)) - k_l}.$$

and

$$F^{a,t}_{n,k_1,\ldots,k_j}(\lambda) = G^{a,t}_{n,k_1,\ldots k_j}(\lambda) \prod_{l=1}^{j} [(n + k_l - \lambda)(ta + (1 - t)(1 - \lambda)) - k_l].$$

Notice that for every value of t with $0 \leq t < 1$, we have that $F^{a,t}_{n,k_1,\ldots,k_j}(\lambda)$ is a polynomial of degree $2j$ and we may analyze its roots as we did the roots of $F_{n,k_1,\ldots,k_j}(\lambda)$.

We define $r_l^{a,t,+}$ and $r_l^{a,t,+}$ to be respectively the largest and smallest roots of the quadratic $(n + k_l - \lambda)(ta + (1 - t)(1 - \lambda))$. As before, we have

$$r_1^{a,t,-} \leq r_2^{a,t,-} \leq \cdots \leq r_j^{a,t,-} \leq n \leq r_j^{a,t,+} \cdots \leq r_1^{a,t,+}.$$

Moreover $G^{a,t}_{n,k_1,\ldots,k_j}(n) > G^{a,t}_{n,k_1,\ldots,k_j}(n) > 0$. Thus we can locate all the roots of $F^{a,t}_{n,k_1,\ldots,k_j}(\lambda)$. It is evident that if k_l appears with multiplicity m in the list $k_1, \ldots, k_l$, then $r_l^{a,t,+}$ and

$r_l^{a,t,+}$ are roots $F_{n,k_1,\ldots,k_j}^{a,t}(\lambda)$ of with multiplicity $m-1$. As long as $l > 1$ and k_l is distinct from k_{l-1} there is a root of $F_{n,k_1,\ldots,k_j}^{a,t}(\lambda)$ between respectively, $r_l^{a,t,-}$ and $r_{l-1}^{a,t,-}$ and between $r_{l-1}^{a,t,+}$ and $r_l^{a,t,+}$. Further, there is a root between $r_j^{a,t,-}$ and n and another root between n and $r_j^{a,t,+}$. We have now accounted for all the roots of $F_{n,k_1,\ldots,k_j}^{a,t}(\lambda)$. In particular, their multiplicities do not depend on t. Therefore, if we denote by $s_k(a,t)$, the kth largest root of $F_{n,k_1,\ldots,k_j}^{a,t}(\lambda)$, we see that $s_k(a,t)$ is real analytic in t for $0 \le t < 1$. (The roots never cross as we vary t between 0 and 1.)

We see moreover that $s_k(a,0) = s_k$, the kth largest root of $F_{n,k_1,\ldots,k_j}(\lambda)$. As t approaches 1, the j largest roots remain bounded below by n and interspersed among the values $r_l^{a,t,+}$. It can be seen that

$$\lim_{t \longrightarrow 1} r_l^{a,t,1} = n + k_l - \frac{k_l}{a}.$$

Thus for $1 \le l \le j$ we have that

$$\lim_{t \longrightarrow 1} s_l(a,t) = s_l(a),$$

where $s_l(a)$ is the lth root of $F_{n,k_1,\ldots,k_j}^{a}(\lambda)$. Since $F_{n,k_1,\ldots,k_j}^{a}(\lambda)$ is a polynomial of degree j, this accounts for all its roots. When $l > j$, we have that

$$\lim_{t \longrightarrow 1} s_l(a,t) = -\infty.$$

Now we prove two sublemmas which combine into the proof of the main lemma. The first relates the roots of certain polynomials $F_{n,k_1,\ldots,k_j}^{a}$ to those of $F_{n,k_1,\ldots,k_j}$ for certain values of a. The second is about the roots of $F_{n,k_1,\ldots,k_j}^{a}$ change as we vary a.

Sublemma 1. *Let $1 \le l \le j$. Then $s_l(1 - s_l) = s_l$.*

Proof of Sublemma 1. . At the point $\lambda = s_l$, there is no difference between $1 - \lambda$ and $1 - s_l$. Thus

$$F_{n,k_1,\ldots,k_j}(s_l) = F_{n,k_1,\ldots,k_j}^{a,t}(s_l) = F_{n,k_1,\ldots,k_j}(s_l) = 0.$$

Since the roots $s_k(a,t)$ never cross as we vary t, it must be that s_l is the lth root of $F_{n,k_1,\ldots,k_j}(\lambda)$. $\square$

Sublemma 2. *We have that*

$$\sum_{l=1}^{j} s_l(a) = jn + \sum_{l=1}^{j} k_l.$$

Further, whenever a and b are negative and $a > b$, we have that

$$F_{n,k_1,\ldots,k_j}^{a} \triangleright F_{n,k_1,\ldots,k_j}^{b}.$$

Proof. The polynomial $F^a_{n,k_1,\ldots,k_j}(\lambda)$ is the determinant of the product

$$
\begin{pmatrix}
a(n+k_1-\lambda) & 1 & \cdots & 1 \\
1 & a(n+k_2-\lambda) & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots \\
1 & 1 & \cdots & a(n+k_j-\lambda)
\end{pmatrix}
\begin{pmatrix}
k_1 & 0 & \cdots & 0 \\
0 & k_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & k_j
\end{pmatrix}.
$$

Thus the roots of the polynomial $F^a_{n,k_1,\ldots,k_j}(\lambda)$ are exactly the spectrum of the matrix

$$
\begin{pmatrix}
n+k_1 & \frac{1}{a} & \cdots & \frac{1}{a} \\
\frac{1}{a} & n+k_2 & \cdots & \frac{1}{a} \\
\vdots & \vdots & \vdots & \vdots \\
\frac{1}{a} & \frac{1}{a} & \cdots & n+k_j
\end{pmatrix}.
$$

The sum of the roots may be calculated by taking the trace. The majorization follows from Corollary 1.3. $\square$

Now we recursively combine Sublemma's 1 and 2 to prove the main lemma. (We repeatedly apply $F^{1-s_l}_{n,k_1,\ldots,k_j}(\lambda) \triangleright F^{1-s_{l-1}}_{n,k_1,\ldots,k_j}(\lambda)$ to the sum of the first $l-1$ roots. Then we use sublemma 1 to identify a root of $F_{n,k_1,\ldots,k_j}(\lambda)$.)

$$
\begin{aligned}
jn + \sum_{l=1}^{j} k_l &= s_j(1-s_j) + s_{j-1}(1-s_j) + \ldots s_1(1-s_j) \\
&= s_j + s_{j-1}(1-s_j) + \ldots s_1(1-s_j) \\
&\geq s_j + s_{j-1}(1-s_{j-1}) + \ldots s_1(1-s_{j-1}) \\
&\geq s_j + s_{j-1} + s_{j-2}(1-s_{j-2}) + \ldots s_1(1-s_{j-2}) \\
&\cdots \\
&\geq s_j + s_{j-1} + \cdots + s_1
\end{aligned}
$$

Which was to be shown.

§5 PROOF OF THE MAIN THEOREM

.

We proceed to prove the Grone-Merris conjecture for 1-regular semibipartite graphs. We prove the nondegenerate case first.

We first observe that $d^T_1 = n + k_1 + \ldots k_j$, that $d^T_l = n$ for $2 \leq l \leq n-1$ and that $d^T_n = j$. Finally, we observe that d^T_l is decreasing and natural number valued. Next we compare

the eigenvalues of L_g. The j largest, all larger than n are the j largest roots of $F_{n,k_1,\ldots,k_j}$, which we call $s_1,\ldots,s_j$. Following close behind are $n-1$ eigenvalues of n coming from the sum zero vectors on the extra vertices. Coming behind then is s_{j+1} which is smaller than j as can be verified by the fact that $G_{n,k_1,\ldots,k_j}(j)$ is positive. All remaining eigenvalues are bounded above by 1.

By the main lemma we have that

$$\sum_{l=1}^{j} s_j \le \sum_{l=1}^{j} d_l^T.$$

This is the jth inequality for the theorem. Since for every $l \le j$ we have $s_l \ge n$, all the lth inequalities follow for $2 \le l \le j$. The first inequality is simply a well known bound on the norm of L_G. The next $n-j-1$ inequalities follow because λ_l and d_l^T both equal n for l in this range. The nth inequality follows because $\lambda_n < j = d_n^T$. The remaining inequalities follow since all other eigenvalues are bounded by 1. As soon some d^T is 0, all later d_T's are zero and the remaining inequalities follow because the final inequality is an equality on trace grounds. We have taken care of the nondegenerate case.

Now we must deal with the degenerate case. This we seem not to have dealt with at all because we derived the polynomial $F_{n,k_1,\ldots,k_j}$ under the assumption of the presence of extra vertices. In fact even the matrix $M(j,k_1,\ldots,k_j)$ does not arise as the action of L_G on functions on the graph constant on the sets W_l. Let us denote the set ofeigenvalues of L_G on that space (with multiplicities) as $\Lambda(j,k_1,\ldots,k_j)$. In fact

$$\Lambda(j,k_1,\ldots,k_j) = \text{spec}(M(j,k_1,\ldots,k_j))\backslash\{j\} = \text{roots}(F_{j,k_1,\ldots,k_j})\backslash\{j\} \cup \{0\}.$$

The last equality follows from continuity of the spectrum.

As is easily verified, $F_{j,k_1,\ldots,k_j}$ has a double root at j corresponding to its jth and $j+1$st roots. One of these is removed to get $\Lambda(j,k_1,\ldots,k_l)$. Now the proof of the first j inequalities in Grone Merris follows just as in the nondegenerate case, and the remaining inequalities are trivial since there is no further eigenvalue greater than 1. The theorem is proved.

References

[DR] Duval, A. and Reiner, V., *Shifted Simplicial Complexes are Laplacian integral*, Trans. AMS **354** (2002), 4313-4344.

[GM] R. Grone and R.Merris, *The Laplacian spectrum of a graph II*, SIAM J. Appl. Math. **7** (1994), 221–229.

[TM] F. Tisseur and K. Meerbergen, *The quadratic eigenvalue problem*, SIAM Review **43** (2001), 235–286.